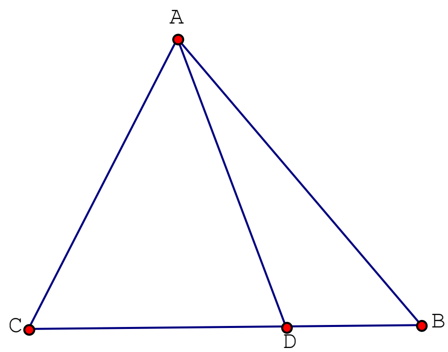
1023. Proposed by George Apostolopoulos, Messolonghi, Greece.

In $\triangle ABC$, $\angle A = 60^\circ$ and $\angle B = 45^\circ$. If D is an interior point on the side BC such that $BD = \frac{BC}{3}$, prove that $\frac{AB^4}{9} + \frac{AC^4}{36} = \tan 15^\circ \bullet AD^4$.

Solution by Arkady Alt , San Jose ,California, USA.



Let $a := BC, b := AC, c := AB, d := AD$. Then identity to prove becomes

$$\frac{c^4}{9} + \frac{b^4}{36} = d^4 \tan 15^\circ \Leftrightarrow$$

$\frac{4c^4 + b^4}{36d^4} = \tan 15^\circ$, where WLOG $a := \sin 60^\circ = \frac{\sqrt{3}}{2}, b := \sin 45^\circ = \frac{1}{\sqrt{2}}$ and

$c := \sin 75^\circ = \cos 15^\circ =$

$\sqrt{\frac{1+\cos 30^\circ}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ and by Stewart Formula $d^2 = \frac{1}{3}b^2 + \frac{2}{3}c^2 - \frac{2}{9}a^2 =$

$\frac{1}{3} \bullet \frac{1}{2} + \frac{2}{3} \bullet \frac{2+\sqrt{3}}{4} - \frac{2}{9} \bullet \frac{3}{4} = \frac{2+\sqrt{3}}{6}$.

Hence, $4c^4 = 4\left(\frac{2+\sqrt{3}}{4}\right)^2 =$

$\frac{7+4\sqrt{3}}{4}, b^4 = \frac{1}{4}, 4c^4 + b^4 = 2+\sqrt{3}, 36d^4 = \left(2+\sqrt{3}\right)^2$ and,

therefore, $\frac{4c^4 + b^4}{36d^4} = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$.

Since $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ then $\sin 15^\circ = \sqrt{1 - \frac{2+\sqrt{3}}{4}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ and, therefore,

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$. Thus, $\frac{4c^4 + b^4}{36d^4} = \tan 15^\circ$.